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13. ABSTRACT (Maximum 200 words) This research seeks to address problems in geometric modeling that require sophisticated and powerful techniques from two traditional mathematical disciplines: differential geometry and algebraic geometry. It also concerns the search for algorithms to effectively implement theoretical results in these disciplines.			
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**Algebro-Geometric and  
Differential Geometric Methods in  
Solid Geometric Modeling**

**FINAL REPORT**  
for the period  
1 December 1987 to 1 March 1992

prepared by

**Dr. Peter F. Stiller  
Professor of Mathematics  
Texas A&M University**

on

23 April 1992

for  
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# FINAL REPORT

by

Dr. Peter F. Stiller

22 April 1992

## §1. Introduction.

This report summarizes the results obtained under U.S. Army Research Office contract DAAL03-88-K-0019 entitled "Algebro-Geometric and Differential Geometric Methods in Solid Geometric Modeling". The principal investigator worked closely throughout the period of performance with researchers at several Army Labs, especially the U.S. Army Ballistic Research Lab. Many of our results have been incorporated into software and are currently in use at these facilities. The principal investigator would like to express his gratitude to a number of researchers at BRL, notably Mr. Edwin O. Davisson and Dr. Paul Deitz, for their support of this research.

As a new and rapidly expanding branch of mathematics, the field of geometric modeling is playing a significant role in the development of improved computer graphics, animation, simulation, and computer assisted design and manufacturing (Mortenson [24]). Increasingly, powerful results and techniques from traditional mathematical disciplines such as differential geometry and/or algebraic geometry are being brought to bear on questions in geometric modeling. Conversely this rich new source of problems is providing new directions and fueling progress in these traditional areas.

Our research sought to address several problems in geometric computation that required sophisticated and powerful techniques from algebraic and differential geometry. Our investigations also included a search for algorithms to effectively implement theoretical results in algebraic geometry. Such algorithms are now a part of what has become known as computational algebraic geometry.

This research originally grew out of the principal investigator's experience at the Ballistic Research Laboratory as a participant in the Army's Summer Faculty Research and Engineering Program. A number of problem areas of mutual interest were identified and formed the basis of our original proposal. Over time the research moved into other areas and the original problem areas generated new questions. These topics and a summary of our results appear below.

## §2. Technical Report.

### §2.1. Problems Studied and Summary of Results.

The specific questions that the principal investigator considered during the course of this research fell into three categories:

1. Differential geometric questions related to computational methods for determining curvature, surface integration techniques, and the tracking of geodesics on surfaces modeled in various ways.
2. Methods for the rapid conversion of a constructive solid geometric model to a boundary representation model, with special attention being paid to the classification and/or explicit determination of surface intersections. This required the application of results and techniques from algebraic geometry and raised a number of issues in computational algebraic geometry.
3. Surface modeling schemes based on rational surface patches.

We treat these sequentially below.

- 1) The extraction of differential geometric information from both CSG (constructive solid geometric) models and surface representations (most notably, non-uniform rational *B*-splines) and the use of that information in analysis (e.g. electro-magnetic signatures, solid mechanics, computational fluid dynamics, etc.) has become crucial to a new generation of computational applications.

Recall that the primary approach to solid geometric modeling at BRL involves constructive solid geometry as opposed to boundary representations. Surfaces and solids are built up from a selected set of primitives which are combined using standard Boolean operations (intersection, union, and complementation). The resulting model is interrogated by ray-tracing. In collaboration with Edwin O. Davisson of the Ballistic Research Laboratory, the principal investigator developed the computational methods for the extraction of metric and second order information from BRL's CSG and *B*-spline models. This is detailed in a BRL technical report, "Curvature and Principal Direction Calculations for MGED Primitives using RT". A copy of the abstract and introduction appears in the appendix.

Among the specific results obtained, we were able to:

- 1) devise a method for computing curvatures and principal directions that is less solid specific and which applies to the broadest possible class of surfaces;
- 2) devise fast and accurate surface integration schemes which make use of curvature information.

To understand 2), recall that typical surface integration schemes often involve facetization of the surface. This amounts to using the best linear approximation to the surface when forming the appropriate Riemann sums. Since the curvature reflects a quadratic approximation, a surface integration scheme based on ray-tracing which makes use of the curvature information computed in 1) provides faster and more accurate integration.

We also investigated accurate methods for tracking geodesics on a modeled surface (primitive or spline based). This has application to computing efficient tool paths for numerically controlled milling. Our investigations centered on the difficult problem of finding geodesics on sufficiently smooth surfaces locally defined by a polynomial

$$f(x, y, z) = 0$$

or on spline modeled surfaces. When a parametric expression is available for the surface locally (e.g. in the spline case) one attempts to numerically solve the system of differential equations which characterize the geodesics. The accuracy of such an approach is the critical issue. We devised algorithms to carry out this procedure, but have yet to implement them.

In the later stages of our research, we choose to focus on the geometric aspects of partial differential equations, particularly adaptive geometry for optimization of physical properties (geometric optimization e.g. minimum radar cross-section). Here the geometric modeling scheme must be coupled and adapted to the partial differential equations and the analysis.

We concentrated on adaptive grid generation and geometric deformation. Current emphasis in adaptive grids, for use with numerical methods to solve partial differential equations, is on elliptic solvers which numerically solve Laplace's or Poisson's equation, given boundary data, in a region. This leads to a nice set of mutually orthogonal grid curves. Unfortunately, the method is computationally intensive and not well suited to a situation where it is necessary to adapt the grid - concentrating grid points in regions where

important physical parameters are rapidly changing and thinning them out elsewhere to increase the accuracy and resolution of the numerical method without loss of computational efficiency. Problems also occur when it becomes important for the ambient geometry to deform in order to find an optimum shape with respect to some physical parameter.

An alternative method is to use *B*-spline parametrizations to provide grids. One previous drawback of this method, when gridding a curved surface or segmenting a curve, was that when you moved the grid points on the surface or along the curve, and then refitted the spline, you destroyed the original surface or curve. We show that this problem can be overcome by “refitting parameter space” (usually  $[0, 1]^n$ ), leaving the original fit alone. The refitting map is given by its graph – an  $n$  dimensional *B*-spline surface in  $[0, 1]^n$ . It then becomes necessary to characterize all *B*-spline maps from  $[0, 1]^n$  to  $[0, 1]^n$  which are 1-1 onto, i.e. what are necessary and sufficient conditions on the control net in the target  $[0, 1]^n$  so as to yield a diffeomorphism of  $[0, 1]^n$ . We were able to prove a theorem in the one dimensional case which gives these conditions – surprisingly the control net need not be monotonic. This result leads to an identifiable space of “grid deformations” on which we can do analysis. For example, mimicking techniques from the theory of harmonic maps, we can produce, via a “principle of least action/minimal energy”, a coordinate system (grid) which reduces the rate of change per grid step of some physical quantity. In essence we achieve a kind of finite-element approach to adaptive grids. Similar techniques can be used to actually deform geometry in response to a physical parameter.

Along similar lines, and for application to the above problem, the principal investigator examined a newly developed technique in non-linear optimization (the so-called equation-based technique). This technique has other interesting applications. One, discussed with Mr. John Grosch of BRL, is to ballistic modeling to optimize gun and shell design. These techniques were designed to apply to complex dynamical system where convergence to a solution must be done concurrent with the optimization in order to stay within computational limits.

2) Methods for rapid conversion from a CSG modeling system to a boundary representation require that special attention be paid to the classification and determination of surface intersections. This remains a difficult problem in geometric modeling. A new type of data

structure, the so-called non-manifold representation (Weiler [32]) is an attempt to combine in one computational paradigm the best features of both CSG systems and surface representation schemes. We investigated this approach and employed techniques from algebraic geometry (intersection theory) to attack problems connected with computationally intersecting geometric objects in several dimensions. Higher dimensional intersection can play an important role in visualization for scientific (e.g. CFD) computation. We also successfully exploited Gröbner basis techniques to speed NURB based intersection algorithms. A preliminary version of our algorithm for intersections has been coded and is undergoing testing. We necessarily were forced to focus on robustness in geometric computation, particularly in regard to our intersection algorithms. Details appear in our paper "Minimal Convex Hulls of Algebraic Varieties with Application to Intersections in Computational Geometry", an abstract of which is attached.

We also investigated contouring algorithms for the fast determination of level curves and surfaces. These curves and surfaces are described locally by equations

$$f(x, y) = 0 \quad \text{or} \quad f(x, y, z) = 0$$

where  $f$  is a multilinear, quadratic, or cubic spline interpolant. The goal is to construct a quick polygonal or facetized approximation to the level set. This requires a complete classification of the topological types of the curve or surface elements that can occur. This has been done. One then needs to insure that as one moves from patch to patch the approximations mesh on the boundary to give a complete description. Some details remain in the higher degree (quadratic and cubic cases), but no problems are anticipated. The final step will be coding the algorithm and testing it.

As mentioned, we were led to consider various aspects of computational algebraic geometry. We proved a nice result that relates volumes in certain "Grassmannians" to surface areas of modeled solids. The result has application to ray-tracing and computing surface areas via ray tracing. An abstract of our paper which presents this result, "Integral Geometry, Grassmannians, and Ray-Tracing", is attached. In addition we investigated algebraic and symbolic tools for the general analysis of solutions to non-linear systems of polynomial equations. Our point of departure was the Generalized Characteristic Polynomial

introduced by Canny, extending the so-called MacCauley and  $U$ -resultants. We developed some additional new perturbation and projection techniques and expect to publish these results.

Finally our work in this area led into some theoretical questions in approximation theory. These questions deal with spaces of splines (finite elements). Consider a surface modeled in one of the above schemes. If we triangulate the surface and study the space of splines of degree  $d$  and smoothness  $r$  on the surface relative to this triangulation, we come up against the problem of determining the dimension of this space and a basis for it. The principal investigator has found that these spaces are related to the cohomology of certain vector bundles – well studied in algebraic geometry. A paper, “Splines and Algebraic Geometry”, detailing this relationship and its import for finite element analysis has been submitted.

3) Surface modeling schemes based on rational surfaces (surfaces that can be parameterized by rational functions, i.e. quotients of polynomials) can combine some of the best features of the CSG and surface representation schemes. Unfortunately, problems of degeneration (surface singularities) can intrude. Understanding the nature of those singularities and how to both control and exploit their presence in the modeling environment is the central problem.

Among the specific questions we considered were:

1. the problem of locally approximating a given curve or surface by a rational curve or surface of a specified degree,
2. the problem of interpolating a “reasonable” set of points in space by passing a rational surface of some specified degree through them.

The main difficulty in solving these problems is that while curves defined by polynomials of degree 1 and 2 are always rational, curves of degree  $\geq 3$  can only be rational if they possess singularities (from the viewpoint of complex projective geometry). Moreover, the generic curve of degree  $d \geq 3$  is non-singular, so we are dealing with a “small” set of curves when we limit ourselves to rational curves. This creates problems when we attempt to interpolate. For example, we can pass a plane cubic  $f(x, y) = 0$  through any nine

points. To see this, recall that such a polynomial involves 10 coefficients, and therefore the space of cubic curves is a 9-dimensional real projective space. Forcing the curve through a point imposes at most one linear condition. However, the space of rational cubics is of lower dimension and it follows that unless the nine points are in special position, we will be unable to pass a rational cubic through them.

This leads to the following questions: Fixing a degree  $d$ , what is the maximum number of points we can take, and still be sure that we can pass a rational curve of degree  $d$  through them? How unique is that rational curve? How is the answer changed if we replace the condition that the curve pass through the set of points by the condition that it pass through certain points with prescribed tangent directions. We made some progress on these questions which are clearly tied to some very deep problems in algebraic geometry.

For surfaces the situation is even more complex. Our paper "On the Arithmetic and Geometry of Elliptic Surfaces" treats some theoretical aspects of the class of so-called rational elliptic surfaces. One particular problem with surfaces is that the locus of singular points can contain entire curves. As these singularities play a significant role in surface geometry, their complexity is an obstacle to modeling. As yet, no satisfactory way has been found to exert sufficient control on the singularities to assure that they do not disrupt our attempt to build a smooth surface out of rational pieces.

Analysis of surface degeneration is the key to understanding how singularities play a central role in the modeling scheme. Because of the complexity involved, symbolic calculation was required to attack some of the questions concerning the classification of specific types of degenerations and the real algebraic geometry of the available rational curves and surfaces which we can use as the building blocks in our modeling scheme. Complete classification even in restricted cases seems out of reach.

On the positive side, surfaces of degree  $\leq 3$  are rational so that, with the exception of the torus and the truncated general cone which are of degree four, one sees immediately that the primitive solids in use at BRL are rational surfaces. In fact we have shown that in the case of BRL's CSG modeling scheme, all of the primitive surfaces are rational (this includes the two quartic surfaces mentioned above), so that exact rational representation is possible.

In a side note, we remark that the line integrals along *rational* curves,

$$\int_C f(x, y, z)dx + g(x, y, z)dy + h(x, y, z)dz$$

where  $f, g, h$  are polynomials, reduce to the form

$$\int_I \frac{p(t)}{q(t)} dt$$

where  $p, q$  are polynomials and that such integrals can be integrated in terms of elementary functions. An analogous statement is true for surface integrals. Thus approximation by rational curves and surfaces can serve as a basis for a line or surface integration scheme.

Finally, we investigated algebraic and symbolic tools for the analysis of solutions to non-linear systems of polynomial equations. Our point of departure was the Generalized Characteristic Polynomial introduced by Canny, extending the so-called MacCauley and  $U$ -resultants. We developed some new perturbation and projection techniques and expect to publish our results shortly.

4) During the course of this research we became interested in the problem of image recognition using range data. Simulated data can easily be obtained by ray-tracing geometric models of the various targets. The necessary geometric and statistical studies can then be performed to test proposed algorithms. The principal investigator has started to develop algorithms based on *wavelets* for use in image recognition problems of this sort.

## §2.2. List of Publications and Technical Reports

The following is a list of papers and technical reports produced during this research effort. Additional related Army technical reports produced outside of this contract, but which made use of the results of this research, are also listed.

### Papers

- 1) P. Stiller, "The Arithmetic and Geometry of Elliptic Surfaces," *Contemporary Math.*, Amer. Math. Soc. (1992), 13 pages, to appear.
- 2) P. Stiller, "Splines and Algebraic Geometry," submitted.
- 3) P. Stiller, "Kloosterman Polynomials and Kloosterman Sums," submitted.
- 4) P. Stiller, "Minimal Convex Hulls of Algebraic Varieties with Application to Intersection to Computational Geometry," preprint to be submitted.
- 5) E. Davisson and P. Stiller, "Integral Geometry, Grassmannians, and Ray-Tracing," preprint to be submitted.

### Technical Reports

- 1) E. Davisson and P. Stiller, "Curvature and Principal Direction Calculations for MGED Primitives using RT," U. S. Army Ballistic Research Laboratory Technical Report.

### Related Technical Reports:

- 1) P. Stiller, "Automatic Target Recognition and 3D Image Generation," Technical Report for the Center for Night Vision and Electro-Optics, Ft. Belvoir.
- 2) P. Stiller, "Splines and Spline Wavelets for TMR," Technical Report to the Instrumentation Directorate, U. S. Army White Sands Missile Range.

For additional details see the abstracts in the appendix.

### **§2.3. Participating Scientific Personnel and Laboratory Contacts**

The principal investigator, Dr. Peter Stiller, and one of his graduate students, Mr. W. Lau, were partially supported by contract funds during the course of this research.

Numerous trips were made to discuss the work with researchers at various Army Labs. Contacts at the U.S. Army Ballistic Research Lab include:

Mr. Edwin Davisson: geometric modelling, image recognition

Dr. Brinton Cooper: algebraic geometry, error correcting codes

Mr. John Gorsch: non-linear optimization, geometric optimization

Mr. Phillip Dykstra: visualization in computation fluid dynamics, geometry of flows

Dr. Kurt Fickie: scientific visualization, wavelets

Dr. Tim Rohaly: scientific visualization, wavelets

The principal investigator also had contacts (as part of other research projects) with individuals at the U.S. Army Center for Night Vision and Electro-Optics at Ft. Belvoir and with the U.S. Army Concepts Analysis Agency. The former in regard to differential geometric aspects of target recognition and the latter in regard to the use of differential geometry, dynamical systems, and stochastic differential equations in combat modeling. A paper "Applications of Differential Geometry to Selected Problems in Combat Analysis" has appeared under CAA cover, and a technical report "Automatic Target Recognition and 3D Image Generation" was delivered to the Center for Night Vision and Electro-Optics. Recently, the principal investigator was in contact with Mr. Elwin Nunn and Mr. Robert Voss at the Instrumentation Directorate at White Sands Missile Range and submitted a report to them on "Spline Methods for Target Motion Resolution". Finally, in November 1991, the principal investigator visited the Army High Performance Computing Research Center in Minneapolis as a Research Fellow.

### §3. Bibliography.

- [1] Bajaj, Hoffman, Hopcroft, Lynch, "Tracing Surface Intersections," Math. Sciences Institute, Technical Report '88-30, Cornell University (1988).
- [2] Bajaj, "Rational hypersurface display," Computer Graphics, Vol. 24, No. 2 (1990).
- [3] Barnhill, "Surfaces in Computer Aided Geometric Design: A survey with new results," Computer-Aided Geometric Design 2 (1985).
- [4] Bartels, Beatty, and Barsky, "An Introduction to Splines for use in Computer Graphics and Geometric Modeling," Morgan Kauffman (1987).
- [5] Bloomenthal, "Polygonalization of implicit surfaces," Computer-Aided Geometric Design 5 (1988).
- [6] Cheng and Goshtasby, "A Parallel  $B$ -Spline Surface Fitting Algorithm," ACM Transactions on Graphics, Vol. 8, No. 1 (1989).
- [7] Coleman, "Adaptive Techniques for Surfaces Grid Generation," Proc. of the Second International Conference on Numerical Grid Generation in CFD (1988).
- [8] Daubechies, "Orthonormal Bases of Compactly Supported Wavelets," Comm. Pure and Applied Math., Vol. 41, No. 7 (1988).
- [9] Daubechies, "The Wavelet Transform: A Method for Time Frequency Localization," preprint.
- [10] DeBoor, "A Practical Guide to Splines," Springer Verlag, Applied Math. Sci. 27 (1978).
- [11] Dennis and Schnabel, "Numerical Methods for Unconstrained Optimization and Non-linear Equations," Prentice Hall Series in Comp. Math. (1983).
- [12] Eiseman, "Adaptive Grid Generation," Computer Methods in Applied Mechanics and Engineering, Vol. 64 Nos. 1-3 (1987).
- [13] Farin, "Curves and Surfaces", Computer-Aided Geometric Design, 2nd ed., Academic Press (1990).
- [14] Fessler, "Nonparametric Fixed-Interval Smoothing with Vector Splines," IEEE Trans. on Signal Processing, Vol. 39, No. 4 (1991).
- [15] Fletcher and McAllister, "Automatic Tension Adjustment for Interpolatory Splines," IEEE Computer Graphics and Applications (1990).

- [16] Heil and Walnut, "Continuous and Discrete Wavelet Transforms," *SIAM Review*, Vol. 31 No. 4 (1989).
- [17] Golub and Van Loan, "Matrix Computations, 2nd ed.," Johns Hopkins University Press (1989).
- [18] Hoffmann, "Geometric and Solid Modeling: An Introduction," Morgan Kauffmann (1989).
- [19] Inselberg, "Multidimensional graphical algorithms and applications," Proc. of the European Computer Conf.-Eurographics '86, North-Holland (1986).
- [20] Lorensen and Cline, "Marching Cubes: A High Resolution 3D Surface Construction Algorithm," *Computer Graphics (ACM Siggraph Proceedings)*, Vol. 21, No. 4 (1987).
- [21] Madych and Nelson, "Multivariate Interpolation and Conditionally Positive Definite Functions I," *Approx. Theory and its Applications* 4 (1988).
- [22] Mallat, "Multiresolution Approximation and Wavelet Orthonormal Bases of  $L(R)$ ," *Trans. AMS*, Vol. 315, No. 1 (1989).
- [23] Mallat, "Multifrequency Channel Decompositions of Images and Wavelet Models," *IEEE Trans. on Acoustics, Speech, and Signal Proc.*, Vol. 37, No. 12 (1989).
- [24] Mortenson, "Geometric Modeling," John Wiley & Sons, New York (1985).
- [25] Mumford and Shah, "Optimal Approximation to Piecewise Smooth Functions and Associated Variational Problems," *Comm. on Pure and Applied Math.*, Vol. XLII (1986).
- [26] Plass and Stone, "Curve Fitting Piecewise Polynomial Cubics," *Computer Graphics SIGGRAPH '83 Conference Proceedings* (1984).
- [27] Potra and Rheinbold, "Differential-Geometric Techniques for Solving Differential Algebraic Equations," Technical Report for the National Science Foundation (1989).
- [28] Quak, Sivakumar, and Ward, "Least Squares Approximation by Radial Functions," preprint.
- [29] Sivakumar and Ward, "On the Least Squares Fit by Radial Functions to Multidimensional Scattered Data," preprint.
- [30] Schmitt et al., "An Adaptive Subdivision Method for Surface Fitting from Sampled Data," *Computer Graphics SIGGRAPH '86 Conference Proceedings* (1986).

- [31] Vanwijk, "New types of Solid Models and their Visualization with Ray-Tracking," Ph.D. Thesis, Technische Hogeschool Delft (Netherlands) (1986).
- [32] Weiler, "Topological Structures for Geometric Modeling," Ph.D. Thesis, Rensselaer Polytechnic Institute (1986).
- [33] Wilhelms and Van Gelder, "Topological Considerations in Isosurface Generation," Computer Graphics, Vol. 24, No. 5 (1990).
- [34] XOX Corporation, "Intersection of Higher Dimensional Shapes for Geometric Modeling," Report on NSF Phase 1 SBIR Award, NSF Grant No. IS185-60193.

#### §4. Appendix: Publications and Abstracts

The following is a list of papers and technical reports produced during this research effort. Abstracts of these items are attached below.

##### Papers

- 1) P. Stiller, "The Arithmetic and Geometry of Elliptic Surfaces," *Contemporary Math.*, Amer. Math. Soc. (1992), 13 pages, to appear.
- 2) P. Stiller, "Splines and Algebraic Geometry," submitted.
- 3) P. Stiller, "Kloosterman Polynomials and Kloosterman Sums," submitted.
- 4) P. Stiller, "Minimal Convex Hulls of Algebraic Varieties with Application to Intersection to Computational Geometry," preprint to be submitted.
- 5) E. Davisson and P. Stiller, "Integral Geometry, Grassmannians, and Ray-Tracing," preprint to be submitted.

##### Technical Reports

- 1) E. Davisson and P. Stiller, "Curvature and Principal Direction Calculations for MGED Primitives using RT," U. S. Army Ballistic Research Laboratory Technical Report.

# The Arithmetic and Geometry of Elliptic Surfaces

PETER F. STILLER

**ABSTRACT.** We survey some aspects of the theory of elliptic surfaces and give some results aimed at determining the Picard number of such a surface. For the surfaces considered, this will be equivalent to determining the Mordell-Weil rank of an elliptic curve defined over a function field in one variable. An interesting conjecture concerning Galois actions on the relative de Rham cohomology of elliptic surfaces is discussed.

This paper focuses on an important class of algebraic surfaces called elliptic surfaces. The results while geometric in character are arithmetic at heart, and for that reason we devote a fair portion of our discussion to those definitions and facts that make the arithmetic clear. Later in the paper, we will explain some recent results and conjectures. This is a preliminary version, the detailed version will appear elsewhere.

There are a number of natural routes leading to the definition of the class of elliptic surfaces. Let  $E$  denote a compact connected complex manifold with  $\dim_{\mathbb{C}} E = 2$ .

**THEOREM 1. (Siegel)** *The field of meromorphic functions on  $E$  has transcendence degree  $\leq 2$  over  $\mathbb{C}$ , i.e. the field of meromorphic functions is:*

- 1)  $\mathbb{C}$  constant functions
- 2) a finite separable extension of  $\mathbb{C}(z)$
- 3) a finite separable extension of  $\mathbb{C}(z, y)$ .  $\square$

Case 3) is precisely the set of algebraic surfaces, i.e. those admitting an embedding into  $\mathbb{P}_{\mathbb{C}}^N$ . Case 2) was studied by Kodaira, leading to a series of three papers:

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1991 Mathematics Subject Classification. 14G10.

The detailed version of this paper will be submitted for publication elsewhere.

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# Splines and Algebraic Geometry

by

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**Abstract:** In this paper we investigate a relationship that exists between certain spaces of splines and the cohomology of certain vector bundles and sheaves on  $\mathbb{P}^n_{\mathbb{C}}$ . We explain the upper-semi continuity of the dimension of certain varying spaces of splines and we give cohomologically based estimates for the generic dimension of such spaces.

## §1. Introduction.

Let  $X \subset \mathbb{R}^d$  be a finite polyhedron which we assume is connected, and let  $\Delta$  be a fixed triangulation of  $X$ . Let  $\{\sigma_1, \sigma_2, \dots, \sigma_r\} \subset \Delta$  be the set of distinct  $d$ -dimensional simplices in  $\Delta$ . We make the following assumptions:

(1.1) 
$$\begin{aligned} 1) \quad & X = \bigcup_{i=1}^r \sigma_i \text{ so that } X \text{ is the union of} \\ & \text{its } d\text{-dimensional simplices,} \\ 2) \quad & \text{for every } i \neq j, \sigma_i \cap \sigma_j = \emptyset \text{ or } \sigma_{ij} \text{ where } \sigma_{ij} \in \Delta \\ & \text{is a } (d-1)\text{-dimensional simplex.} \end{aligned}$$

It is easy to see that  $X$  has the topological type of a  $d$ -manifold with boundary. In particular, the boundary  $\partial X$  has the topological type of a compact  $(d-1)$ -manifold (not necessarily connected).

Our principal object of study will be the finite dimensional vector space  $S_m^r(\Delta)$  of piecewise polynomial functions of degree  $\leq m$  on  $X$ , which possess continuous derivatives to order  $r$ . Specifically,  $f \in S_m^r(\Delta)$  if and only if

$$1) \quad f \in C^r(X)$$

and

$$2) \quad f|_{\sigma_i} = p_i$$

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\* Partially supported by U.S. Army Research Office Grant DAAL03-88-K-0019.

# On Kloosterman Polynomials and Kloosterman Sums\*

Peter F. Stiller

**Abstract:** This paper explores some properties of Kloosterman polynomials, including symmetry properties and coefficient estimates. Their relationship with Kloosterman sums and to questions of the distinctness of Kloosterman sums is examined.

Let  $p$  be a prime and  $\mathbb{F}_q$  be a finite field with  $q = p^f$  elements. Consider a non-trivial additive character  $\psi: \mathbb{F}_q \rightarrow \mathbb{C}^\times$ , say  $x \mapsto e^{2\pi i Tr(x)/p}$  where  $Tr = Tr_{\mathbb{F}_q/\mathbb{F}_p}$  is the trace. Recall the definition of the Kloosterman sums

$$Kl(q, a) = \sum_{x \in \mathbb{F}_q^\times} \psi\left(x + \frac{a}{x}\right)$$

for  $a \in \mathbb{F}_q^\times$ .

For this family of exponential sums, parameterized by  $a \in \mathbb{F}_q^\times$ , we can ask if the values  $Kl(q, a)$  are distinct. Posed this way the question has a negative answer. If we denote by  $\sigma$  the Frobenius element of  $Gal(\mathbb{F}_q/\mathbb{F}_p)$ , so that  $\sigma(x) = x^p$ , then it is easy to see that

$$Kl(q, a) = Kl(q, \sigma a).$$

So instead, we can ask if, apart from this action of  $Gal(\mathbb{F}_q/\mathbb{F}_p)$  on  $\mathbb{F}_q^\times$ , the values  $Kl(q, a)$  are distinct.

The first partial result is due to B. Fisher [7]:

**Theorem 1:** For the prime field  $\mathbb{F}_p$ , the Kloosterman sums

$$Kl(p, a) = \sum_{x \in \mathbb{F}_p^\times} \psi\left(x + \frac{a}{x}\right) = \sum_{xy=a} \zeta_p^{x+y}$$

for  $a \in \mathbb{F}_p^\times$  are distinct.

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\* AMS subject classification: 11Lxx

This paper is in final form and no version of it will be submitted for publication elsewhere.

Minimal Convex Hulls of Algebraic Varieties with Application to  
Intersection in Computational Geometry

by

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**Abstract:** Given positive integers  $d, n, r$  and  $k$  with  $r < n$ ,  $n < k$ , we investigate sets of  $k$  points in  $\mathbb{R}^n$  whose convex hull contains an algebraic patch of degree  $d$ ,  $[0, 1]^r \rightarrow \mathbb{R}^n$ , and has minimal volume. We also show that the control points of Bezier cubics have this property. Applications of the results to the computation of geometric intersections are discussed.

**Key Words:** Convex hull, algebraic variety, Bezier curves, Bezier surfaces, geometric computation.

**AMS Subject Classification:** 68U05 and 52A99.

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# Integral Geometry, Grassmannians, and Ray-Tracing

by

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**Abstract:** We show that the measure of certain weighted sets of directed lines can be used to determine the surface area of a suitably regular solid in  $\mathbb{R}^3$ . The results are then applied to devise ray-tracing and sampling methods for various computations.

## §1. Introduction.

There are two major approaches to modeling geometric objects in three-dimensional space. One method focuses on boundary representations which describe only the oriented surface (boundary) of the solid object. The other method is known as constructive solid geometry (CSG). It makes use of a set of simple geometric primitives, which are combined using Boolean operations (union, intersection, and difference), to construct complex three-dimensional objects.

In the typical CSG scheme, ray-tracing is used to interrogate and/or render the geometry. This means that one usually has at one's disposal the ability to "sample" the geometry by firing randomly selected rays through it. In this paper we shall examine this sampling process and the geometric information that can be extracted from a solid object in this "probabilistic" way. The mathematical discipline used to address questions of this sort is known as Integral Geometry.

**Key Words:** Invariant metric, integral geometry, sampling, ray-tracing, geometric modeling, Grassmannians.

**AMS Subject Classification:** 68U05.

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## Curvature and Principal Direction Calculations for MGED Primitives using RT

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### ABSTRACT

An introduction to the basic concepts of curvature and related topics is given. Specific methods used in version 1.2 of the RT ray-tracing program for the calculation of principal curvatures and principal directions are presented.

### INTRODUCTION

Curvature is a classical notion in differential geometry. Loosely speaking, it is a measure of the bending of a curve or surface situated in space. Unlike tangent lines or tangent planes which involve first order data and are the best linear approximation to a curve or surface at a point, curvature involves second order data.

This report begins with a review of the basic concept of curvature and several related ideas that the reader will need in order to follow the later calculations. It then examines in detail the methods used to implement the calculation of curvatures and principal directions in version 1.2 of the Ballistic Research Laboratory's RT code. The final section details the calculations specific to each primitive and provides full documentation of the code. Appendix A contains a discussion of eigenvalue calculations pertinent to the curvature and principal direction determination. A method for computing the curvatures and principal directions for spline surfaces is included in appendix B.